On plane flow of a gas with finite electrical conductivity in a strong magnetic field

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The principal object of study is plane flow over bodies with a sharp apex at Mach numbers greater than unity. The magnetic field is assumed to be uniform, rectilinear, and parallel to the undisturbed stream. Flow behaviour near the apex of a wedge is investigated by the method of characteristics. It is found that for small wedge angles an attached shock attenuates initially with distance from the apex, but for larger wedge angles the shock grows stronger.

The structure of a slow magneto-gasdynamic shock is investigated for the case of strong magnetic field and small electrical conductivity. The streamlines are displaced within the shock although the initial and final flow directions are the same. An ordinary gasdynamic shock may occur on the upstream side of the transition. The shock structure theory gives a solution for the flow near the apex of a certain class of bodies.

For the study of slow shock structure, it is shown that the transition is described by a curve in the (F, H)-plane. F is the sum of pressure and momentum flux in the direction of variation; H is the sum of enthalpy and kinetic energy due to the velocity component in the direction of variation. General properties of the (F, H)-plane are found for a gas whose equation of state obeys the conditions of Weyl (1949). Flow behaviour on the transition curve is then determined. The theory of the (F, H)-plane can be used in the study of other one-dimensional processes in magneto-gasdynamics.

1. Introduction

The present paper discusses some aspects of the steady flow of a gas in the presence of a strong magnetic field. Conditions approach the formal limit

$$b/a \to \infty \quad \text{with} \quad v/a = O(1), \tag{1}$$

where b is the Alfvén wave speed, a the sound speed and v the gas velocity. The effects of viscosity, heat conduction and radiation are neglected, but those of electrical resistance are not.

We shall consider plane two-dimensional flow over a body with a sharp apex. The upstream flow is assumed to be uniform, have a Mach number greater than unity, and be parallel to a uniform magnetic field. We shall in fact consider a magnetic field which is uniform throughout the flow, justifying the assumption by the following arguments. The limit (1) implies that the magnetic pressure, $B^2/2\mu$ in M.K.S. units, approaches an infinite value in comparison to the pressure or momentum of the gas. If the magnetic field were to be perturbed significantly by currents within the gas, magnetic forces would be of the order of changes in magnetic pressure. There appears to be no means by which such forces could be supported, and so we assume that gas currents can be neglected when determining the form of the magnetic field. In any flow problem the magnetic field can be considered as known. It must have the form of a free-space field, and in particular it could be uniform and rectilinear.

For a steady plane flow where the free stream is parallel to the magnetic field, the electric field is zero. Currents are induced in the gas only when there is motion across the magnetic field lines. The resulting magnetic forces tend to resist this motion with an effectiveness measured by the magnetic force coefficient

$$C_M = \sigma B^2 l/\rho v, \tag{2}$$

where σ, l and ρ are typical values of the electrical conductivity, length scale and density. When $C_M \to 0$, magnetic forces are negligible in comparison to typical inertial forces, and flow behaviour is similar to that of ordinary gasdynamics (non-conducting flow). Thus for $C_M \to 0$, it is known that there are realistic steady-state solutions with the condition of uniform upstream flow which we have assumed. In the present work we shall be considering flow over bodies for which there is an attached shock at the apex when $C_M \to 0$. For small but non-zero values of C_M it is reasonable to suppose that the flow pattern is not greatly affected, and flow deflexion at the apex is still given by a shock.

However, an attached shock is not likely to occur for all values of C_M. When $C_M \to \infty$, motion across the field lines must be reduced to a negligible drift for the scale considered, and the gas is effectively channelled by the magnetic field. A body will act as a stopper to a channel with the same thickness as the body thickness, provided that there are no currents within the body to perturb the field. The gas in the channel must be at rest relative to the body. As will be clearer from the discussion of wave motion given below, the starting process to achieve this type of flow could be readily given by the one-dimensional propagation of shock waves in the magnetic channel. When C_M is large but still finite, the drift velocity across the channel must be of order $1/C_M$ times the free-stream velocity, and mass conservation requires the gas to be virtually at rest for a distance of order C_M times the body thickness. Clearly there is some restriction, as yet unknown, on the value of C_M for which an attached shock is possible. This point will be discussed again at the end of the paper.

It will be noted that the magnetic Reynolds number R_M has not been mentioned so far. The arguments for assuming a free-space form for the magnetic field rested on a discussion of typical forces, and remain valid whatever the value of R_M. However, equation (2) may be written

$$C_M = R_M b^2/v^2,$$

and, with $R_M \neq 0$, $C_M \to \infty$ in the limit given by equation (1). Thus the flow should be effectively channelled by the magnetic field for all non-zero values of R_M, and the parameter is not significant. This point is of interest since flow experiments at high R_M are difficult to achieve, whereas a reasonable range of

C_M values are available under typical conditions obtained in a combustion-driven shock tube. The present study in fact arises out of experiments on the flow over axially symmetric bodies which are being made in a shock tube. Strictly the limit (1) cannot be approached very closely under the experimental conditions, but it does constitute a useful idealization which may lead to a better understanding of more general flows. In the same spirit we have chosen a plane geometry here rather than the analytically more difficult one of axial symmetry. An approximate analysis for the flow near the vertex of a cone with an attached shock and with similar upstream conditions has been given by Barthel & Lykoudis (1961). Their approximations were based on the hypersonic assumption of a thin shock layer, but a more general analysis is readily given for the flow near the apex of a wedge, and this is done in §4. Another idealization of the present work is the neglect of Hall effect, although in practice it is difficult to reproduce strong-field conditions where this neglect is justified.

The strong-field limit (1) has been presented in terms of wave speeds, but so far the significance of the wave speeds as such is not obvious. The behaviour of waves in a gas of infinite conductivity has been studied by Lighthill (1960) for the limit $b/a \to \infty$. Of the three magneto-gasdynamic modes, the Alfvén mode is not significant with the assumed plane geometry since it requires transverse field and velocity components. The fast mode proceeds with speed b in all directions and carries magnetic and electric field perturbations. The gas must move with the magnetic field, but under strong-field conditions the effects of gas inertia are only important for large accelerations, and hence the speed of propagation is high. The slow mode proceeds one-dimensionally along the field lines at the sound speed a. Lighthill interpreted the slow mode as the propagation of sound waves in rigid magnetic channels, magnetic forces strong enough to prevent lateral motion always being possible. Lighthill also suggested that disturbances created by a body moving at moderate speeds would be propagated by the slow mode only.

With finite conductivity, we may expect a more rapid spread of magnetic-field perturbation in the fast mode, since the gas is not so strongly tied to the field lines. In the slow mode there will be some propagation of disturbances perpendicular to the field lines. Consider a particular starting process which might lead to the type of steady flow that we are assuming. The gas is at rest relative to the body, and the magnetic field is uniform and steady. Motion of the gas is initiated by the passage of a strong shock in the direction of the field lines, e.g. shock-tube flow. The magnetic field will remain effectively unperturbed throughout the starting process for the limit $b/a \to \infty$, provided that there is no mechanism external to the gas that will affect the field directly, e.g. a time-varying current in an external conductor. Transient electric fields could have an important influence in general, but they may be taken as effectively irrotational during the starting process. In the present case, the electric field will be zero throughout, provided that the flow remains plane, and the boundary conditions on the electric field are such that no potential differences can be supported. Then it will be necessary only to consider a modified slow-mode propagation.

With zero electric field and constant magnetic field, Ohm's law shows that the current density, and hence the magnetic force density, at a point will depend only on the gas velocity and conductivity at that point. The magnetic force term in the equation of motion does not contain velocity gradients, and thus electromagnetic effects are localized during the starting process. Although disturbances in the slow mode may spread in a direction perpendicular to the field lines when there is finite conductivitiy, the basic mechanism for their propagation will still be the sound-wave mechanism of ordinary gasdynamics. A weak front may be attenuated by the action of magnetic forces, but it will proceed at speed a. From these arguments it appears that the range of influence of a point in steady continuous flow should be bounded still by the Mach lines through the point. The higher the conductivity the weaker the influence becomes in a direction perpendicular to the field lines, but the range remains the same. Only when the influence is reduced to negligible proportions with high conductivity can we consider the range to be altered, i.e. to a strip of infinitesimal thickness aligned with the field but lying downstream of the point. Thus the significance of the sound speed is in some respects similar to its significance in ordinary gasdynamics.

Throughout the following work we shall always refer to the flow Mach number being greater or less than unity when $v > a$ or $v < a$. The terms subsonic and supersonic are used in a special sense in §§5 and 6.

There is one further simplification to the flow problem with the conditions we have assumed. When the electric field is zero, the stagnation enthalpy is constant along streamlines, heat conduction, radiation and viscous effects being neglected. Since the upstream state is assumed to be uniform, we have

$$\operatorname{grad}(h + \tfrac{1}{2}v^2) = 0, \tag{3}$$

where h is the enthalpy.

2. Thermodynamic relations

Given equation (3) and thermodynamic equilibrium, it is convenient to have an equation of state in the form

$$p = f(h, \rho). \tag{4}$$

Introducing the thermodynamic property γ_h, defined by

$$a^2 = (\partial p/\partial \rho)_s = \gamma_h (\partial p/\partial \rho)_h, \tag{5}$$

we can obtain the differential form of equation (4) as

$$dp = \frac{\gamma_h - 1}{\gamma_h} \rho \, dh + \frac{a^2}{\gamma_h} d\rho, \tag{6}$$

γ_h is related to the ratio of specific heats γ_T, and the isentropic index γ by

$$\gamma_h (\partial p/\partial \rho)_h = \gamma_T (\partial p/\partial \rho)_T = \gamma p/\rho. \tag{7}$$

When p/ρ and h are functions of the temperature T only, we have $\gamma_h = \gamma_T = \gamma$. In general,

$$\gamma_h - 1 = -\frac{a^2}{\rho}\left(\frac{\partial \rho}{\partial h}\right)_p = \frac{1}{\rho T}\left(\frac{\partial p}{\partial s}\right)_\rho > 0, \tag{8}$$

if the conditions of Weyl (1949) are assumed.

For the ideal partially ionized gas discussed by Goldsworthy (1961)

$$\gamma_h - 1 = \frac{4\xi(1-\xi)\,TT_i + \{8+10\xi(1-\xi)\}\,T^2}{4\xi(1-\xi)\,(T_i^2+3TT_i) + \{12+15\xi(1-\xi)\}\,T^2}, \tag{9}$$

where ξ is the degree of ionization, and T_i is the characteristic temperature of ionization. Typical values on a combustion-driven shock tube with Argon as the working gas are $\xi = 0{\cdot}12$, $T/T_i = 0{\cdot}06$, and $\gamma_h - 1 = 0{\cdot}056$.

For the purposes of some calculations we shall take the limit $T/T_i \to 0$, $\xi \neq 0$. The appropriate equation of state is given by

$$(p/\rho)_{T/T_i \to 0} \to \text{const.}, \tag{10}$$

since it is readily shown that h, and hence ξ and T, remain constant. The limit can be taken formally from more general analysis by setting $\gamma_h = 1$, $a^2 = \text{const.}$ In the context of shock-tube flows the adoption of the above simple (p, ρ)-relation implies that the energy equation is no longer strictly satisfied, but the approximation is hardly worse than that of neglected radiation.

3. Vorticity creation at shock waves

An ordinary gasdynamic shock with no change in magnetic field across it is a possible flow feature if R_M and C_M, with length based on shock thickness, tend to zero. The change in flow velocity will in general imply a sudden change in the current density and hence in the magnetic force on the gas. The action of the magnetic force can be such as to impart rotation to fluid elements, and vorticity is created.

Suppose that the co-ordinate system (x_s, y_s, z) has the x_s-axis coinciding locally with an outward normal on the downstream side of a shock, the z-direction being perpendicular to the flow and magnetic field plane. From the momentum equation for an inviscid fluid we have

$$\rho v_x[\omega_z] + [\partial p/\partial y_s + \rho v\, \partial v/\partial y_s] = B_x[j_z], \tag{11}$$

where the square brackets denote the jump in the quantity enclosed, ω_z is the vorticity, and we have used the fact that mass flow is conserved. The second term on the left-hand side of equation (11) is familiar in ordinary gasdynamics. Here and in the subsequent analysis, the subscripts x and y are used to denote vector components in the x and y directions for the co-ordinate system under consideration; the extra subscripts on x and y, which are used to distinguish particular co-ordinate systems, are dropped.

For a uniform flow parallel to a uniform magnetic field upstream of the shock, equation (11) gives on the downstream side

$$\omega_z = \pm \frac{B_0}{\rho_0 v_0} j_z - \frac{1}{\rho v_x}\left(\frac{\partial p}{\partial y_s} + \rho v \frac{\partial v}{\partial y_s}\right),$$

where B_0, ρ_0 and v_0 are the magnitudes of the field, density and velocity on the upstream side of the shock. Using the result of Hayes (1957) for the creation of vorticity due to the second term on the right-hand side of the above equation, we obtain

$$\omega_z = \pm \frac{B_0}{\rho_0 v_0} j_z + v_0 \frac{(1-\epsilon)^2}{\epsilon} \cos\beta_0 \frac{d\beta_0}{dy_s}, \tag{12}$$

where ϵ is the density ratio ($\epsilon < 1$) and β_0 is the angle the shock makes with the upstream flow. With a shock attached at the apex of a body, the vorticity creation of equation (12) represents the initial effect of the magnetic field. If the body is wedge shaped, $d\beta_0/dy_s$ will be non-zero at the apex unless the voriticity can be accommodated by a purely rectilinear shear flow in that region. In the next section it will be shown that there are flows for which $d\beta_0/dy_s = 0$, but in general shocks are curved at the apex of a wedge.

4. Flow at the apex of a wedge

We shall investigate the flow near the apex of a wedge by constructing an elementary characteristic net. A direct approach to the problem can be made from the intrinsic flow equations, as in the analysis of shock curvature in terms of streamline curvature in ordinary gasdynamics. The author believes that the present method is more plausible since it focuses attention on the way in which a point on the wedge behind the apex can influence flow at the shock.

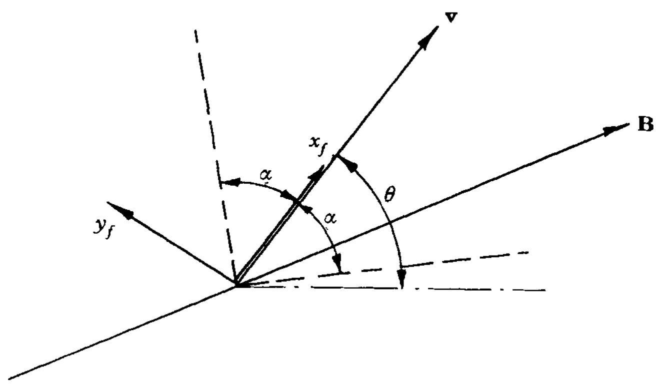

FIGURE 1. Intrinsic co-ordinate system.

However, it is not obvious that the final algebraic results can be extended directly to flow with Mach number less than unity behind the shock if there are no singularities in the flow. It may be verified that the results can be so extended.

If (x_f, y_f, z) are the intrinsic co-ordinates at a point, so that x_f is directed along the streamline (figure 1), the momentum and continuity equations are

$$\rho v(\partial v/\partial x_f) + (\partial p/\partial x_f) = -j_z B_y, \tag{13}$$

$$\rho v^2(\partial \theta/\partial x_f) + (\partial p/\partial y_f) = j_z B_x, \tag{14}$$

$$\partial \rho v/\partial x_f + \rho v(\partial \theta/\partial y_f) = 0, \tag{15}$$

where θ is the angle between the streamline and a fixed direction. Ohm's law gives

$$j_z = \sigma v B_y. \tag{16}$$

By means of equations (3), (6) and (13) the first term of equation (15) can be eliminated and we obtain

$$\rho v^2(\partial \theta/\partial y_f) + (M^2 - 1)(\partial p/\partial x_f) = \{(\gamma_h - 1) M^2 + 1\} j_z B_y, \tag{17}$$

where $M = v/a$ is the flow Mach Number. For the limit $b/a \to \infty$, B_x and B_y can be considered as known, and the magnetic force terms depend directly on flow properties and not on their derivatives. The equation for any one property must then be hyperbolic when $M > 1$ and have the same characteristic directions as in ordinary gasdynamics. The corresponding equations for axially symmetric flow were studied by Hains, Yoler & Ehlers (1960). Equation (13) gives directly

$$\frac{\sin^2\alpha}{\gamma}\frac{dp}{p}+\frac{dv}{v} = -\frac{j_z B_y}{\rho v^2}\mathrm{d}x_f \quad \text{on} \quad \frac{dy_f}{dx_f}=0, \tag{18}$$

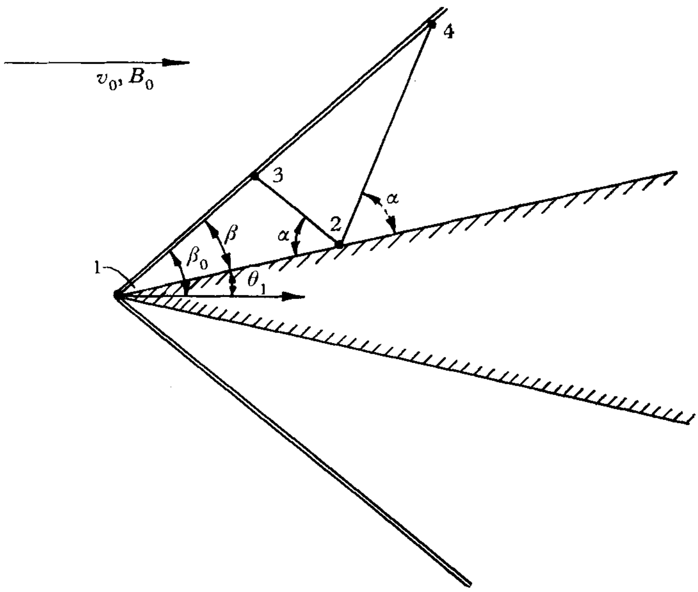

FIGURE 2. Characteristic net near the apex of a wedge.

where α is the Mach angle and γ is the isentropic index of equation (7). Multiplying equation (14) by dx_f, equation (17) by dy_f and adding, we obtain

$$\pm\frac{\sin 2\alpha}{2\gamma}\frac{dp}{p}+\mathrm{d}\theta = \frac{j_z}{\rho v^2}(B_x dx_f + B_y dy_f + (\gamma_h - 1)M^2 B_y dy_f) \quad \text{on} \quad \frac{dy_f}{dx_f} = \pm\tan\alpha. \tag{19}$$

In general the terms due to the magnetic forces will only be small for a scale such that C_M is small.

Suppose that 1 (see figure 2) is a point at the apex of a wedge immediately behind an attached shock. The characteristics with directions given by equation (19) at a point 2 on the wedge surface meet the shock at points 3 and 4 (also shown in figure 2). Let the extent of the region 124 be small so that C_M based on a typical length tends to zero. To zero order in C_M the shock and characteristics are straight, and the flow is uniform. To first order in C_M we may apply equation (19) to the zero-order characteristics and obtain

$$\begin{aligned}\{(p_4-p_3)/2\gamma_1 p_1\}\sin 2\alpha_1 + (\theta_4-\theta_1)+(\theta_3-\theta_1) = (j_{z_1}B_0/\rho_1 v_1)[\delta_{24}\{&\cos(\alpha_1+\theta_1)\\ -(\gamma_{h_1}-1)\sin\theta_1\operatorname{cosec}\alpha_1\} - \delta_{32}\{&\cos(\alpha_1-\theta_1)\\ +(\gamma_{h_1}-1)\sin\theta_1\operatorname{cosec}\alpha_1\}],&\end{aligned} \tag{20}$$

where θ is measured to the free-stream direction, B_0 is the magnitude of the magnetic field, and δ_{24}, δ_{32} are the lengths of the characteristics. To this order of approximation the magnitude of the perturbations varies directly with the length scale, and hence θ varies linearly along the shock. We can thus relate $(\theta_4-\theta_1)$ and $(\theta_3-\theta_1)$ to $(\theta_4-\theta_3)$ and the geometry of the figure writing

$$[\{(dp/d\theta)_s/2\gamma_1 p_1\}\sin 2\alpha_1+\tan\alpha_1\cot\beta_1](d\theta/dl)_s\,\delta_{34}$$
$$=j_{z_1}B_0/\rho_1 v_1^2[\delta_{24}\{\cos(\alpha_1+\theta_1)-(\gamma_{h_1}-1)\sin\theta_1\operatorname{cosec}\alpha_1\}$$
$$-\delta_{32}\{\cos(\alpha_1-\theta_1)+(\gamma_{h_1}-1)\sin\theta_1\operatorname{cosec}\alpha_1\}],$$

where β is the downstream shock angle, $(dp/d\theta)_s$ gives the initial variation of p with θ behind the shock for this angle, and $(d\theta/dl)_s$ the variation of θ with length l along the shock.

δ_{34}, δ_{24}, and δ_{32} can be related by the geometry of the figure. It is convenient to define a non-dimensional length scale in terms of the upstream properties (subscript 0) so that length along the shock is given in terms of L_s, where

$$L_s=\sigma_0 B_0^2 l/\rho_0 v_0. \tag{21}$$

Then, substituting for j_z from equation (16), we obtain

$$[\{(dp/d\theta)_s/2\gamma p\}\sin 2\alpha+\tan\alpha\cot\beta]\,d\theta/dL_s$$
$$=-(\sigma/\sigma_0)\sin\beta\sin\theta\operatorname{cosec}(\beta+\theta)$$
$$\times[\sin(\alpha+\beta)\operatorname{cosec}2\alpha\{\cos(\alpha+\theta)-(\gamma_h-1)\sin\theta\operatorname{cosec}\alpha\}$$
$$-\sin(\alpha-\beta)\operatorname{cosec}2\alpha\{\cos(\alpha-\theta)+(\gamma_h-1)\sin\theta\operatorname{cosec}\alpha\}], \tag{22}$$

where the subscript 1 has been dropped for the state immediately behind the shock. Thus, given sufficient information on shock properties, we may calculate $d\theta/dL_s$ at the apex of a wedge and hence determine the shock curvature $d\beta_0/dL_s$ where β_0 is the shock angle relative to the upstream flow. Further analysis readily gives the pressure and, with equation (18), the velocity variation on the wedge surface. A similar method can be used when the flow upstream of the shock does not conduct electricity and the magnetic field is no longer parallel.

When $\theta\to 0$, the shock is weak and $\beta\to\alpha\to\alpha_0$. Using the known Mach wave relation for $(dp/d\theta)_s$ we obtain

$$2(d\theta/dL_s)_{\theta\to 0}\to\theta\cos\alpha_0.$$

For linearized flow over a slender wedge, the effect of magnetic forces is small up to moderate values of L_s, and we may integrate. Thus the flow angle immediately behind a Mach wave is given by

$$\theta=\theta_1\exp(-\tfrac{1}{2}L_s\cos\alpha_0). \tag{23}$$

The wave decays exponentially, and weak shocks will tend to curve towards the wedge. With increasing θ, the contribution to equation (22) from the 32 characteristic increases and the contribution from the 24 characteristic can change sign. Thus for a given upstream flow state there will at least be one wedge angle for which $d\theta/dL_s=0$, and flow behind the shock is initially rectilinear. It is easily verified that vorticity in this case corresponds to that given by equation (12) when there is zero shock curvature. If we assume that the 'Crocco' point of ordinary gasdynamics occurs when the flow Mach number behind the shock is less than unity, the term in brackets on the left-hand side of equation (22) is positive while characteristics exist. Then for large wedge angles $d\theta/dL_s$ becomes positive and the shock curves away from the wedge. The analysis of

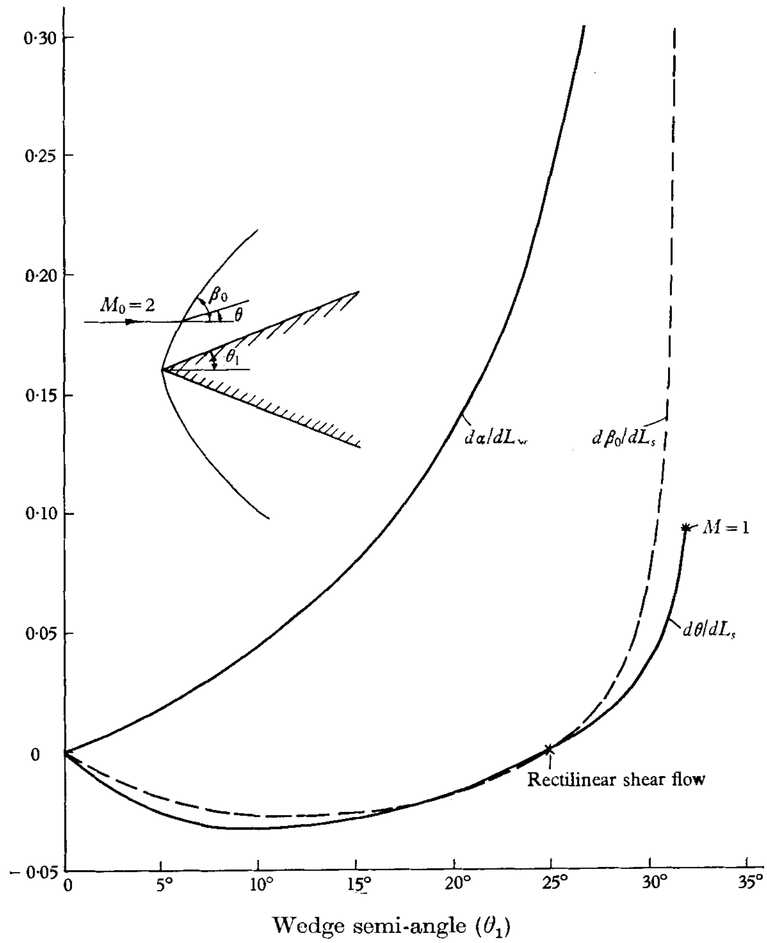

FIGURE 3. Variation of properties at the apex of a wedge.

Barthel & Lykoudis (1961) showed a similar result for strong shocks in the flow over cones.

Some calculations have been made for a gas with an equation of state given by equation (10), p/ρ constant, with σ constant. The variation of $d\theta/dL_s$ and $d\beta_0/dL_s$ with wedge angle for a free-stream Mach number of 2 is shown in figure 3. Also included in figure 3 is the rate of change of Mach angle with length along the wedge surface, $d\alpha/dL_w$, where L_w has the same non-dimensional form as L_s. The flow Mach number decreases along the wedge, and for a considerable range of wedge angles this is a more significant effect than the shock curvature.

In general, the condition for zero shock curvature in terms of the free-stream properties is found to have the simple form.

$$\cos 2\beta_0 = \tfrac{1}{2}(\cos 2\alpha_0 - 1), \tag{24}$$

when the equation of state has the form p/ρ constant.

5. The strong-field shock

To investigate the subsequent development of the flow behind an attached shock, we take a particular case of the inverse problem where the shock wave position is assumed and the body shape is to be determined. Suppose that the shock is plane. For a shock-oriented co-ordinate system (x_s, y_s, z), we have $\partial/\partial y_s = 0$, and the subsequent flow is one-dimensional After the initial deflexion of the flow, the effect of the magnetic force must be to re-align the streamlines with the field lines since there can be no pressure gradient to balance forces in the y_s-direction. The flow behaviour is indicated schematically in figure 4. A particular streamline is chosen to define the body shape on one side of the apex. A mirror image completes the flow pattern for a symmetrical body.

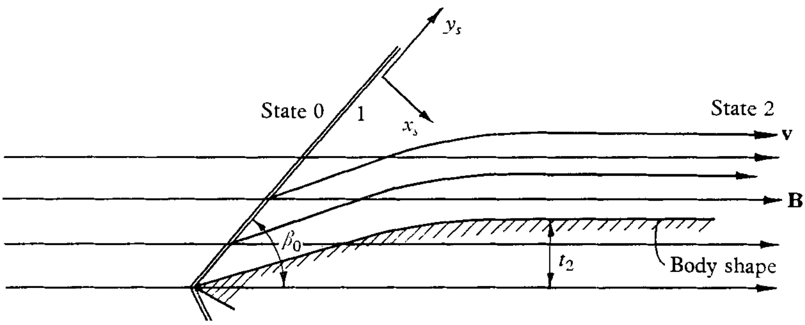

FIGURE 4. The strong-field shock.

Let the upstream flow be at state 0, the flow immediately after the shock be at state 1, and state 2 occur where the flow and magnetic field are again parallel. Mass conservation between 0 and 2 requires that

$$\rho_0 v_0 = \rho_2 v_2. \tag{25}$$

By conservation of momentum in a direction parallel to the magnetic field lines (i.e. perpendicular to the magnetic force), we have

$$\rho_0 v_0^2 + p_0 = \rho_2 v_2^2 + p_2. \tag{26}$$

Thus, with constant stagnation enthalpy, states 0 and 2 are related in the same way as the states on either side of a shock in ordinary gasdynamics which is normal to the stream. It then follows that the Mach number at state 2 is less than unity. We shall show that it is not necessary for the transition to be initiated by a shock, and flow from state 0 to state 2 is always possible whatever angle the x_s-direction makes with the field lines provided the Mach number at state 0 is greater than unity. The transition 02 is in fact a slow magnetogasdynamic shock in the limit $b/a \to \infty$, and the flow pattern indicated in figure 4 is a typical shock structure for weak electrical conductivity.† The Mach number conditions discussed above are equivalent to the usual conditions for the ratio of normal velocity components to slow wave speed.

We shall refer to the complete transition 02 as 'the strong-field shock' in contradistinction to the ordinary gasdynamic shocks. The one-dimensional process which forms the structure is referred to as the 'strong-field Ohmic dissipation process'. It is related to processes in ordinary gasdynamics studied by Shercliff (1958). We shall use the same symbolism as Shercliff and quote freely from his results. The terms subsonic and supersonic will refer to states where $v_x < a$ and $v_x > a$, respectively.

Conservation of mass momentum, and energy within the strong field shock gives

$$\rho v_x = G, \tag{27}$$

$$\rho v_x^2 + p = F = F^0 - (B_y/B_x)\,Gv_y, \tag{28}$$

$$h + \tfrac{1}{2}v_x^2 = H = H^0 - \tfrac{1}{2}v_y^2, \tag{29}$$

where G, F^0 and H^0 are constants fixed by the end states. For thermodynamic equilibrium the entropy variation is related to variations in F and H by

$$T\,ds = dH - dF/\rho. \tag{30}$$

The strong-field Ohmic dissipation process has F, H and s all varying, but by equations (28) and (29)

$$(F^0 - F)^2 = 2G^2(B_y/B_x)\,(H^0 - H), \tag{31}$$

which is a parabola in an (F, H)-plane. Given values of G, F and H may define the end states of an ordinary gasdynamic shock with one state subsonic and the other supersonic. Alternatively, they may define a single subsonic state or no state. Each point on the parabola can not correspond to more than two flow states. In the subsequent discussion we shall indicate the extent of the regions of the (F, H)-plane where two states, one state or no state are possible. Figure 5 illustrates these properties.

It is well known (e.g. see Shercliff 1958) that H has its maximum value in a Rayleigh-line process (G and F constant) at the sonic point $v_x = a$, and only one such point occurs. Similarly for the Fanno line (G and H constant) F is a minimum at $v_x = a$. The locus of sonic states in the (F, H)-plane then represents a boundary below and to the right of which no flow state is possible. From equations (28), (29), (6) and with G constant the slope of the sonic line is given by

$$(\partial F/\partial H)_{v_x = a} = (\gamma_h - 1)\,\rho/\gamma_h. \tag{32}$$

For a perfect gas the right-hand side of equation (32) is readily found to be proportional to $1/F$, so that the sonic line is then parabolic.

If we assume $p \to 0$ and $h \to 0$ represents the upper limit of supersonic states, equations (27), (28) and (29) give for this limit

$$(F^2)_{p,h\to 0} \to 2G^2H, \tag{33}$$

† This structure has been shown by Kulikovskii & Liubimov (1961), but their results are mistakenly presented as being true for Ohmic diffusion small in comparison to viscous diffusion instead of the reverse.

which is again parabolic, and will be termed the hypersonic line. The slope of the hypersonic line can be expressed as

$$(\partial F/\partial H)_{p,\, h\to 0} = \rho. \tag{34}$$

We take the limit of subsonic states to be given by $v_x \to 0$, $\rho \to \infty$. If $h \to 0$ as $\rho \to \infty$ with p finite, the F-axis forms part of the subsonic limit, and, if $p \to \infty$ as $\rho \to \infty$ with h non-zero, the remainder of the limit is given as $F \to \infty$ for all H.

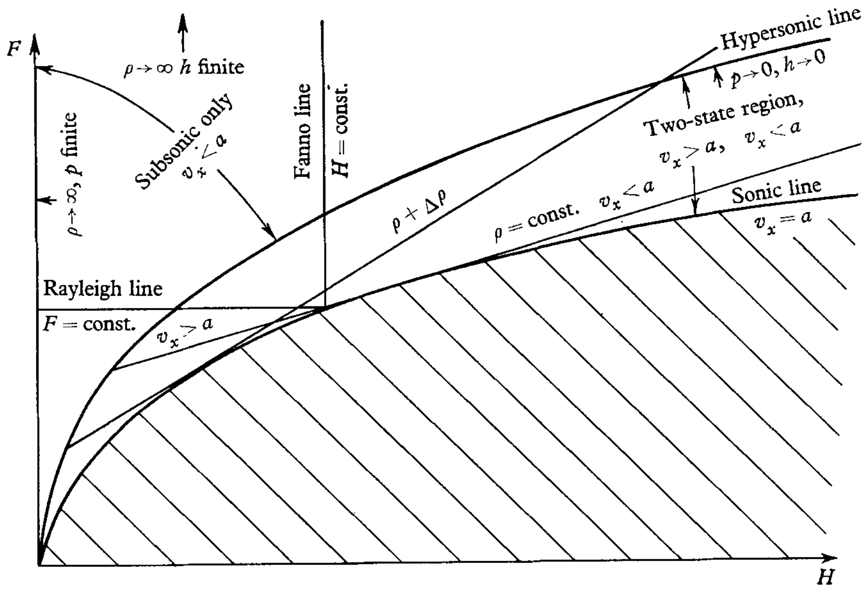

FIGURE 5. The (F, H)-plane for one-dimensional processes with G = const.

Lines of constant density and hence of constant v_x have a slope given by

$$\left(\frac{\partial F}{\partial H}\right)_\rho = \left(\frac{\partial p}{\partial h}\right)_\rho = \frac{\gamma_h - 1}{\gamma_h}\rho. \tag{35}$$

They each touch the sonic line once only, provided that the thermodynamic state is uniquely determined by ρ and a. The sonic line is their envelope. Following a line of constant ρ as H and F increase, comparison of equations (34) and (35) shows that the line starts at the hypersonic limit and traverses a series of supersonic states until the point of contact with the sonic line is reached. Beyond the sonic line the states are subsonic. The direction of increasing ρ in the (F, H)-plane can be inferred from the known properties of the Fanno line:

$$(\partial F/\partial \rho)_{GH} > 0 \quad \text{when} \quad v_x < a, \qquad (\partial F/\partial \rho)_{GH} < 0 \quad \text{when} \quad v_x > a.$$

For perfect gases equation (35) shows that the lines of constant density are straight.

It is worth noting that there are many one-dimensional processes in magnetogasdynamics which can be easily represented in the (F, H)-plane. For example, the general Ohmic-dissipation line where conditions are the same as in the present case except that the limit $b/a \to \infty$ is no longer taken and B_y may vary is found to be also parabolic. Channel flow where $B_x = 0$, $B_y = \text{const.}$ (strong-field assumption), $v_y = 0$, and the electric field has a uniform z-component, is represented by a straight line.

Flow behaviour on the strong-field Ohmic-dissipation parabola is most easily discussed in terms of entropy and the quantity

$$K = v_y B_x / v_x B_y, \tag{36}$$

which measures how far the direction of the streamlines diverges from that of the field lines. The slope of the Ohmic-dissipation line can be expressed as

$$(\partial F/\partial H)_{\text{Ohmic}} = \rho/K. \tag{37}$$

The following argument will depend largely on a study of points where the Ohmic-dissipation line cuts one of the limit lines of the (F, H)-plane. Conditions are imposed by the relative slopes of the lines. For easy reference table 1 summarizes equations (32), (34), (35) ad (37) and figure 6, where some completed forms of the Ohmic-dissipation line are shown, may be also consulted.

Slope	Ohmic-dissipation line	Sonic line $v_x = a$	Hypersonic line $p \to 0,\ h \to 0$	Constant density line
$\dfrac{\partial F}{\partial H}$	ρ/K	$\dfrac{\gamma_h - 1}{\gamma_h}\rho$	ρ	$\dfrac{\gamma_h - 1}{\gamma_h}\rho$
	$\left(K = \dfrac{v_y B_x}{v_x B_y}\right)$	$(\gamma_h - 1 > 0)$		

TABLE 1

From equations (30) and (37), we have

$$T(\partial s/\partial H)_{\text{Ohmic}} = (1 - 1/K). \tag{38}$$

Then s is stationary when $K = 1$ or when H is stationary provided that $K \neq 0$. At a point where the Ohmic-dissipation line cuts the sonic line, there can be a continuous succession of states from subsonic to supersonic with H, and hence s, stationary where $v_x = a$. H has a maximum value when $H = H^0$, but $K = 0$, and in general s is not stationary. There can not be more than two states at which $K = 1$, the end states of a strong field shock. It is easily shown that

$$\left(\frac{\partial H}{\partial K}\right)_{\text{Ohmic}} = \frac{v_x^2 + v_y^2 - a^2}{v_y^2(a^2 - v_x^2)}, \quad \text{where} \quad K = 1. \tag{39}$$

The nature of the entropy extrema can be established from equation (39), table 1, and general properties of the end states of strong field shocks. A summary is given in table 2. The nature of difficult points, as when $v_x = a$ and $K = \gamma_h/(\gamma_h - 1)$, can usually be inferred from the geometry of the (F, H)-plane.

On the Ohmic-dissipation line, states where $F > F^0$ have $v_y < 0$. Since $v_{y0} > v_{y_2} > 0$, we have $F_0 < F_2 < F^0$. Suppose state 2 is given and consider the range $F_2 < F < F^0$. The Ohmic-dissipation line must lie above the sonic line for the whole of this range, for, if it did not, an intersection with H maximum, s

maximum, and a continuous succession of subsonic states from $F = F_2$ to the point of intersection must occur. Since s already has a maximum at state 2, and there can be no intermediate point where s is a minimum, the intersection is not possible, and there is a continuous succession of subsonic states between $F = F_2$ and $F = F^0$. Since $K = 0$ at $F = F^0$ and $K = 1$ at state 2, we have $K < 1$ for the whole range, and there will be some subsonic points in the neighbourhood of $F = F_2$, but $F < F_2$, where $K > 1$. At a point on the Ohmic-dissipation line where there is a supersonic state, the density and hence K are smaller for the supersonic state than for the subsonic state. Hence for supersonic states in the range $F_2 < F < F^0$, we have $K < 1$, and by table 1 the slope of the Ohmic-dissipation line at any hypersonic state is greater than the slope of the hypersonic line. It follows that there can be just one hypersonic state if state 2 lies below the hypersonic line, and the point where $F = F^0$ lies above it.

Consider now the range $F \leqslant F_2$, and suppose that the Ohmic-dissipation line cuts the sonic line. If there is more than one intersection, let the one with the greatest value of F be at $F = F_s$. There can only be an intersection with the hypersonic line for the range $F_s < F < F_2$ if state 2 lies above it, since both lines are parabolic and have parallel axes. Then at the hypersonic state $K < 1$. There is a continuous variation of K from $K > 1$ to $K < 1$ following the subsonic states in

$v_x = a$	$\left(\frac{\partial F}{\partial H}\right)_{\text{Ohmic}} > \left(\frac{\partial F}{\partial H}\right)_{v_x=a}$,	$K < \frac{\gamma_h}{\gamma_h - 1}$	$\begin{Bmatrix} H \text{ min} \\ F \text{ min} \end{Bmatrix}$	s min., $K > 1$ s max., $K < 1$
	$\left(\frac{\partial F}{\partial H}\right)_{\text{Ohmic}} < \left(\frac{\partial F}{\partial H}\right)_{v_x=a}$,	$K > \frac{\gamma_h}{\gamma_h - 1}$	$\begin{Bmatrix} H \text{ max} \\ F \text{ max} \end{Bmatrix}$	s max.
	$v_x > a$,	state 0	—	s max.
$K = 1$	$v_x < a$, $v_x^2 + v_y^2 > a^2$,	state 0	—	s min.
	$v_x < a$, $v_x^2 + v_y^2 < a^2$,	state 2	—	s max.

TABLE 2. Entropy extrema on the Ohmic-dissipation line, $K > 0$

the neighbourhood of state 2 to sonic at $F = F_s$ and the supersonic states back to the hypersonic line or to $F = F_2$. Hence state 0, where $K = 1$, must lie in the range $F_s < F < F_2$. At $F = F_s$, we have $K > 1$ if state 0 is supersonic, so that from table 2, s is a minimum. The result is consistent with s being a maximum at state 2 and a maximum at state 0. Similarly, if state 0 is subsonic, we have $K < 1$, and s is a maximum at $F = F_s$.

Suppose that the Ohmic-dissipation line does not cut the sonic line at all, but does cut the hypersonic line (necessarily at two points). At one intersection we have $K > 1$ and at the other $K < 1$ by table 1. Between the intersections there is a range of supersonic states at one of which $K = 1$ for state 0.

If the Ohmic-dissipation line lies wholly above the hypersonic line, we only have subsonic states to consider. As $H \to 0$ for F^0, we have $K \to \infty$. Hence we have $K > 1$ in the whole range of possible states with $F < F_2$. There can be no state 0. The limit to possible strong field shocks is given when the Ohmic-dissipation line just touches the hypersonic line. At the point of contact $K = 1$ for the hypersonic state (table 1) so that state 0 lies on the hypersonic line and state 2 above it.

The possible forms of the Ohmic-dissipation line when the end states of a strong field shock occur are shown in figure 6. The cases where there is no intersection with the sonic line (figure 6*a*) and where state 0 is subsonic (figure 6*c*) necessarily occur in practice. The case where there is an intersection with the sonic line but state 0 is supersonic (figure 6*b*) appears to be reasonable by virtue of the following argument. The strong field shock could have state 0 exactly sonic. The Ohmic-dissipation line then cuts the sonic line at a non-zero angle.

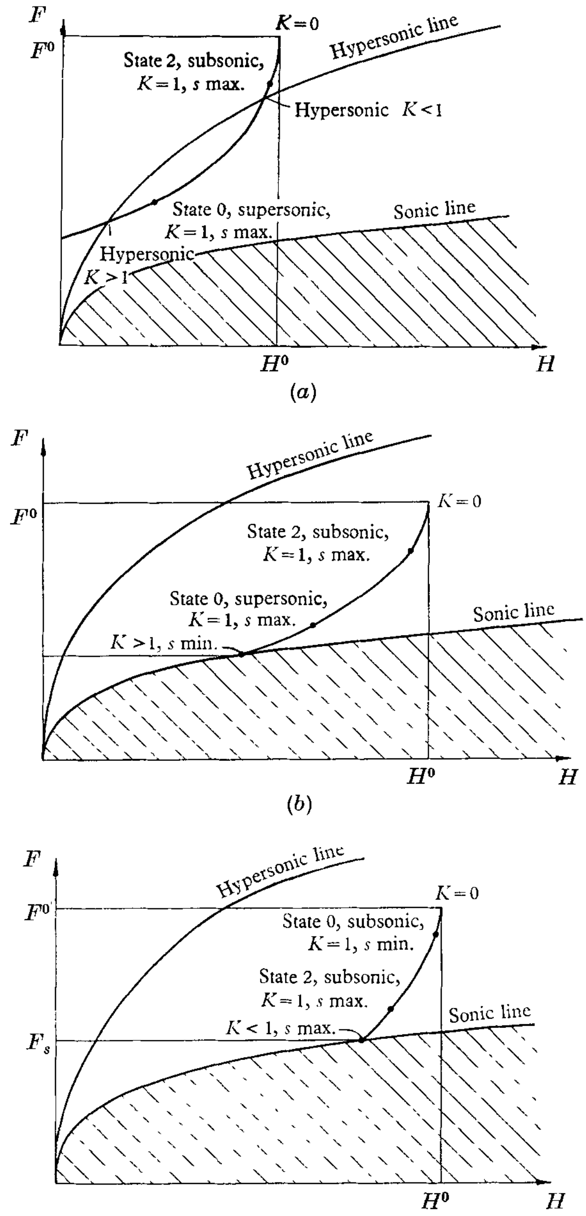

FIGURE 6. Strong-field shock transitions on the Ohmic-dissipation line.

A small perturbation of state 0 so that v_{x0}/a increases will not alter the position of the Ohmic-dissipation line significantly, and the intersection will remain.

For Ohmic dissipation the entropy increases in the direction of positive mass flow. This fact provides a criterion by which possible transitions can be determined. If state 0 is subsonic, one-dimensional flow from 0 to 2 via the subsonic states is possible. If state 0 is supersonic, s has a maximum at state 0. However, an initial transition to the subsonic state 1 at $F = F_0$, $H = H_0$, can be made by an ordinary gasdynamic shock, and a continuous one-dimensional flow from state 1 to state 2 is then possible. Thus for the present purposes it is only the flow behaviour at the subsonic states of the Ohmic-dissipation line which is of interest. However, it is worth noting that flow on other parts of the strong-field Ohmic-dissipation line could be realized in channel flow. Also, if a one-dimensional flow is initiated by an 'ionizing' shock with a non-conducting upsteam flow, the initial conditions could be very different from those of a strong-field shock.

5. General properties of the strong-field shock transition

For the one-dimensional flow on the Ohmic-dissipation line which forms part or all of the strong field shock structure F increases and hence by equation (28) v_y decreases continuously. The variation of v_x and hence of ρ is not easily determined. The difficulty arises from the fact that the Ohmic-dissipation line may touch lines of constant density, and points where ρ is stationary occur.† For a perfect gas the lines of constant density are straight, and the sonic line is parabolic in the (F, H)-plane. Two common tangents to the Ohmic-dissipation line and the sonic line are possible for the configuration of figure 6*a*. The points of contact with the Ohmic-dissipation line define a point of minimum density for a subsonic state and a point of maximum density for a supersonic state provided that the point lies below the hypersonic line. At these points we have $K > 1$ by table 1, so that the minimum occurs where $F < F_2$ and the maximum where $F < F_0$. That the density minimum certainly occurs within some strong field shock transitions can be inferred from the case where state 0 lies on the hypersonic line. For perfect gases the density ratio across the initial gasdynamic shock is the same as the density ratio across the whole strong field shock. That it does not always occur for the configuration of figure 6*a* is easily inferred from the case where the Ohmic dissipation line touches the sonic line.

Figures 7*a*, *b* and *c* show the variation of v_y with v_x for the transitions of figures 6*a*, *b* and *c*. We have taken the typical density variation of a perfect gas. Only the transitions which begin with a gasdynamic shock are of interest in the context of flow over bodies with a sharp apex.

An indication of the streamline shape, and hence of the body shape is given by K, and lines of constant K in the (v_y, v_x)-plane are straight lines through the origin. The body thickness increases with distance from the apex. If we take the case of a very weak initial gasdynamic shock, it is clear that $(\partial K/\partial v_y) < 0$ in the vicinity of state 1, and the slope of the body surface increases initially with distance from the apex, a result which may be compared with that of §4 where a

† The author is indebted to Dr J. A. Shercliff for showing that these points exist.

weak shock at the apex of a wedge tends to curve towards the body surface. When the initial shock is almost normal to the stream, state 1 is close to state 2, $(\partial K/\partial v_y) > 0$, and the slope of the body decreases continuously.

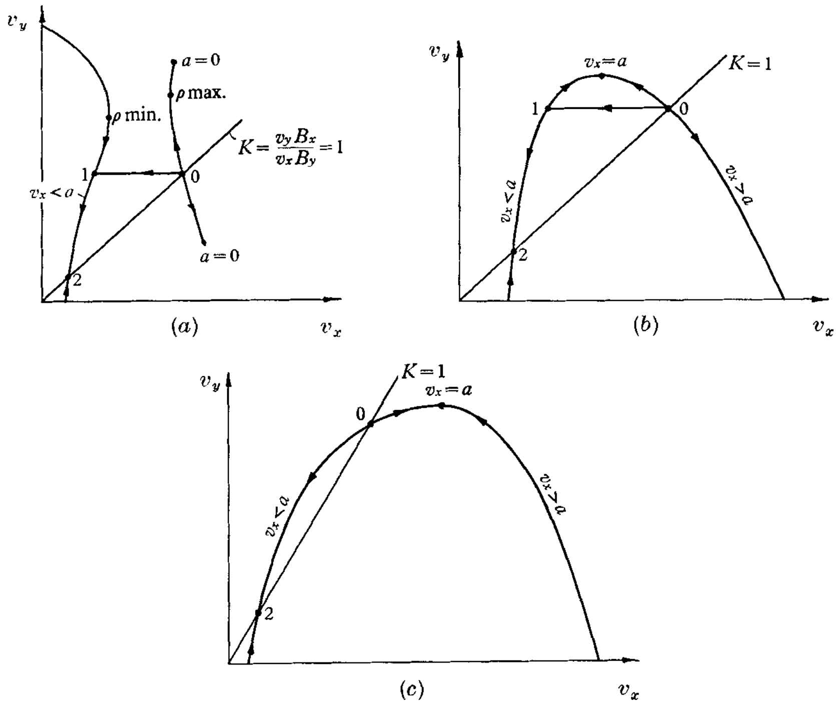

FIGURE 7. Strong field shock transitions variation of v_y with v_x. The arrows indicate the direction of increasing entropy.

The momentum equation for the y_s direction is

$$G(dv_y/dx_s) = j_z B_x. \tag{40}$$

Ohm's law can be used to eliminate j_z from equation (40), and in principle the resulting equation may be integrated to find the detailed flow behaviour in the physical plane. Clearly C_M based on shock thickness is of order unity.

The currents within the strong field shock transition are equivalent to a sheet current with density given by

$$J = \int_0^2 j_z dx_s.$$

Under purely one-dimensional conditions this current gives rise to a slight perturbation of the field at state 2 if b/a is large but not infinite. The field is deflected towards the x_s direction, a result familiar in the theory of slow shocks. Using equation (40) we can obtain

$$\left|\frac{\mu J}{B_y}\right| = \frac{v_0^2}{b_0^2}(1/\epsilon - 1), \tag{41}$$

where ϵ is the density ratio of a gasdynamic shock normal to the free stream. However, in the context of flow over bodies with a sharp apex we may note that field perturbations will extend upstream of an attached shock to satisfy field conditions at the body.

For a gas with p/ρ constant, as in equation (10), and with constant electrical conductivity the strong-field shock analysis is greatly simplified. We shall summarize the more interesting results. The general shape of the Ohmic-dissipation line in the (v_y, v_x)-plane is always similar to that of figures 7*b* and *c*. There are no points at which the density is stationary. There is just one point at which K is stationary, and K then has a maximum value. This point may be within a strong field shock transition, and $v_x^2 + v_y^2 > a^2$ there. The slope of a body

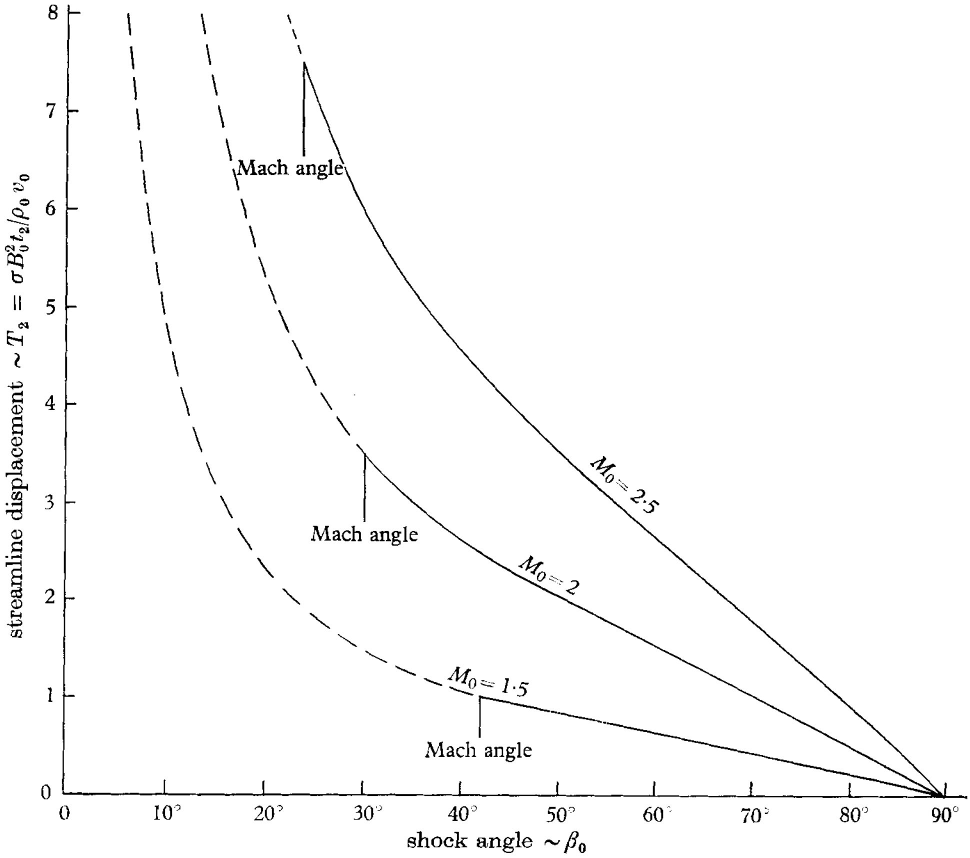

FIGURE 8. Displacement of the streamlines in strong field shocks.

surface shaped to fit the streamlines of a strong field shock increases continuously to the point where maximum K is reached and then decreases if the initial gasdynamic shock is weak. The slope of the body surface decreases continuously if the initial gasdynamic shock is strong. With K stationary at state 1, the body surface has zero curvature initially, and the appropriate condition is the same as that for zero shock curvature at the apex of a wedge, i.e. equation (24).

The shock thickness is found to be logarithmically infinite when p/ρ is constant, but the displacement of the streamlines in a direction perpendicular to the field lines t_2, is in general finite. For a symmetrical body t_2 is half the thickness of a body shaped to fit the streamlines of two strong field shocks (see figure 4). Some results for t_2 have been computed and are shown in figure 8. It is convenient to use a non-dimensional thickness parameter, given by $T_2 = \sigma B_0^2 t_2/\rho_0 v_0$, where values at state 0 are used. T_2 is plotted against strong field shock angle β_0 for given upstream Mach number M_0. When β_0 is less than the upstream Mach angle, there is no initial gasdynamic shock, and it is found that $T_2 \to \infty$ as $\beta_0 \to 0$, the velocity in a direction perpendicular to the field lines being reduced to a small drift.

T_2 is of order M_0^2 since a more natural value of density in the thickness parameter would be $\rho_2 = \rho_0/M_0^2$. The strong field shock thickness is of order t_2/M_0^2, so that the concept of a thin shock layer on the forward part of a body shaped to fit the streamlines of a strong-field shock is valid for $M_0^2 \to \infty$. Note that there is a pressure difference across the Newtonian layer, and the body has the form of a wedge initially.

6. Discussion of the flow over bodies with a sharp apex

In the previous sections we have seen how the strong field shock gives a solution to the body-shape problem for a plane gasdynamic shock at the apex. However, the results do not imply that, given an appropriate body shape, the strong field shock solution occurs in practice. The following physical arguments suggest that the solution is not likely to be realistic. The strong-field shock can extend an infinite distance in a lateral direction, and flow behaviour in the outer regions should approach the typical behaviour expected for infinite conductivity. As mentioned in § 1, Lighthill (1960) showed that the propagation of disturbances is then effectively that of sound waves in rigid magnetic channels. Thus in any starting process we would not expect a build up of strong perturbations in the outer flow regions by virtue of disturbances originating at the body. On the whole it seems reasonable to suppose that there will be a tendency towards reduction in lateral disturbance for the magnetogasdynamic flows in comparison to flows in ordinary gasdynamics.

Consider now a body which terminates in a flat base. Between apex and base the body surface is chosen to fit the streamlines of part of the strong field shock solution. As the position of the base approaches the apex C_M, based on maximum body thickness, tends to zero. In the limit the flow pattern will be the same as in ordinary gasdynamics and under suitable conditions could have a Mach number greater than unity throughout. At least for low values of C_M, it is probable that the Mach number could remain greater than unity. We are then justified in taking part of the strong field shock analysis for a solution to the hyperbolic flow over the forward part of the body, since sufficient boundary conditions are satisfied. The influence of the sharp corner provided by the base cannot extend further upstream than the first Mach line emanating from the corner. Hence a part of the strong field shock solution is valid for a region bounded by this Mach line, the body surface, and the initial gasdynamic shock attached at the apex. An investigation of the subsequent flow pattern is beyond the scope of this paper, although in principle the method of characteristics may be used. In the immediate neighbourhood of the corner the flow behaviour will tend to that of a Prandtl–Meyer expansion, since the scale may be reduced to a point where the magnetic forces can be neglected in describing the behaviour. However, without a full treatment of the subsequent flow we cannot say for certain at what value of C_M the type of flow pattern described might break down. Clearly an upper limit to this value of C_M is given by the condition that the Mach number immediately before the corner is just unity with the assumed flow pattern. For greater values of C_M the condition that the Mach number remains unity at the corner, as in transonic theory of ordinary gasdynamics, will certainly require a radical change.

As a final calculation for a gas with p/ρ constant, we take the case of a shock angle of 52° 14′ with an upstream Mach number of 2. The initial streamline curvature of a stong field shock is then zero. At the point where the Mach number is unity the streamline displacement t is given by $\sigma B_0^2 t/\rho_0 v_0 = 0{\cdot}64$. The inclination of the streamlines to the free stream direction is 24° 53′ at the apex (see figure 3) and 22° 6′ at the point where the Mach number is unity. In this case the body shape is very nearly that of a wedge over the whole range for which the strong field shock solution is possibly valid. This does not imply that a constant-area stream-tube approximation should provide a reasonable basis for calculation since small changes of area can have a significant effect when the Mach number is close to unity.

In the present paper we have explored some aspects of flow in which a shock remains attached at the apex of a body It is clear that the value of C_M, based on body thickness, cannot exceed a certain value for any particular body shape, if the assumption of shock attachment is to be still valid. Although the study may begin to give some idea of the values of C_M for which radical changes in flow pattern may be expected, no definite criteria have emerged. It is hoped that further discussion of these problems will be stimulated. For the corresponding case of axially symmetric flow there is some hope of experimental verification, and the break-down of a certain type of flow pattern should be easily observed.

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